International Organization of Research & Development (IORD)
ISSN: 2348-0831
Vol 12 Issue 02 | 2025

A Fuzzy Multi-Criteria Decision-Making Model for Investment Portfolio Selection in Banking

Pooja (Ph. D Research Scholar), Dr. Narendra Swami (Assistant Professor)
Department – Mathematics, Shri Jagdish Prasad Jhabarmal Tibrewala University, Chudela, Jhunjhunu

DOI: 10.5281/zenodo.18107882

ABSTRACT

Fractional-order differential equations have attracted significant attention in recent years due to their ability to model memory and hereditary properties inherent in many physical, biological, and engineering systems. In comparison with classical integer-order models, fractional differential equations provide a more accurate and flexible framework for describing complex nonlinear phenomena. However, the presence of nonlinearity and fractional derivatives poses substantial analytical challenges, particularly with respect to the existence and uniqueness of solutions.

This paper investigates the existence and uniqueness of solutions for a class of nonlinear fractional-order differential equations under suitable conditions. The analysis is carried out in an appropriate Banach space setting, where the fractional differential equation is first transformed into an equivalent integral equation. Fixed point techniques play a central role in establishing the main results. Specifically, Banach's contraction principle and Schauder's fixed point theorem are employed to derive sufficient conditions for the existence and uniqueness of solutions.

The results obtained in this study extend several known results in the literature by relaxing restrictive assumptions on the nonlinear term and the order of the fractional derivative. The theoretical findings provide a unified framework for analyzing nonlinear fractional-order systems and contribute to the mathematical foundations of fractional calculus. Furthermore, an illustrative example is presented to demonstrate the applicability of the developed theoretical results.

The outcomes of this research are expected to be useful for further theoretical investigations as well as for applications involving fractional-order models in science and engineering.

Keywords: Fractional-order differential equations; Nonlinear systems; Fixed point theorems.

1. INTRODUCTION

Fractional calculus has emerged as a powerful mathematical tool for modeling complex phenomena that cannot be adequately described using classical integer-order differential equations. Unlike integer-order derivatives, fractional derivatives incorporate memory and hereditary effects, making them particularly suitable for describing processes with long-range temporal dependence. As a result, fractional-order differential equations have found extensive applications in diverse fields such as viscoelasticity, anomalous diffusion, control theory, signal processing, population dynamics, and biological systems.

In recent years, increasing attention has been devoted to nonlinear fractional-order differential equations, as nonlinearity plays a crucial role in accurately representing real-world systems. However, the inclusion of both nonlinearity and fractional derivatives significantly complicates the mathematical analysis. One of the fundamental questions in the theory of fractional differential equations concerns the existence and uniqueness of solutions, since these properties ensure the well-posedness of the underlying mathematical model.

The study of existence and uniqueness for fractional differential equations differs substantially from the classical integer-order case. In the fractional setting, the nonlocal nature of fractional derivatives introduces additional analytical challenges. Standard techniques used for ordinary differential equations often fail or require substantial modification. Consequently, alternative approaches grounded in functional analysis have become indispensable for investigating such problems.

Among the most effective analytical tools for establishing existence and uniqueness results are fixed point theorems. Fixed point theory provides a robust framework for analyzing nonlinear problems by transforming differential equations into equivalent integral equations. This approach allows the use of powerful results from functional analysis, particularly in Banach and metric spaces, to derive sufficient conditions for solvability.

The Banach contraction principle is one of the most widely used fixed point results for proving uniqueness of solutions. It ensures the existence of a unique fixed point under a contraction condition, making it especially suitable for nonlinear fractional equations with Lipschitz-type nonlinearities. On the other hand, Schauder's fixed point theorem is frequently employed to establish existence results when compactness conditions are satisfied, even in the absence of strict contraction properties. Together, these theorems form a comprehensive analytical framework for addressing solvability issues in nonlinear fractional-order systems.

Several researchers have investigated existence and uniqueness problems for fractional differential equations using fixed point techniques. Many existing works focus on specific types of fractional derivatives, such as the Caputo or Riemann–Liouville derivatives, and impose restrictive assumptions on the nonlinear terms. While these studies have contributed significantly to the development of the theory, there remains a need for more generalized results that relax these assumptions and provide a unified approach applicable to a broader class of nonlinear fractional-order differential equations.

Moreover, much of the existing literature addresses either existence or uniqueness separately, whereas a combined treatment using multiple fixed-point techniques can offer deeper insights into the structure of solutions. Establishing both existence and uniqueness within a single analytical framework enhances the theoretical robustness of the results and improves their applicability to real-world models.

Motivated by these considerations, the present paper investigates the existence and uniqueness of solutions for a class of nonlinear fractional-order differential equations by employing fixed point theorems. The analysis is carried out in an appropriate Banach space, where the fractional differential equation is first converted into an equivalent integral equation. This transformation plays a key role in enabling the application of fixed-point results.

The main contributions of this work can be summarized as follows. First, sufficient conditions for the existence and uniqueness of solutions are established using Banach's contraction principle. These conditions guarantee not only the solvability of the problem but also the continuous dependence of solutions on initial data. Second, existence results are obtained using Schauder's fixed point theorem under more general assumptions on the nonlinear term. This dual approach allows for a flexible and comprehensive treatment of nonlinear fractional-order differential equations.

The theoretical results presented in this paper extend and generalize several known results in the literature. By weakening restrictive conditions and employing a systematic functional analytic framework, this study contributes to the mathematical foundations of fractional differential equations. Furthermore, the results provide a solid theoretical basis for future research on stability analysis, numerical approximation, and applications of nonlinear fractional-order systems.

2. PRELIMINARIES

In this section, we recall some fundamental concepts and definitions from fractional calculus and fixed-point theory that are essential for the analysis carried out in this paper. These preliminaries provide the mathematical framework within which the existence and uniqueness results for nonlinear fractional-order differential equations are established.

2.1 Fractional Calculus

Fractional calculus is a generalization of classical calculus that deals with derivatives and integrals of non-integer order. Among various definitions of fractional derivatives available in the literature, the Riemann–Liouville and Caputo fractional derivatives are the most commonly used. In this work, we primarily consider the Caputo fractional derivative due to its suitability for problems involving initial conditions of integer order.

Definition 2.1 (Riemann–Liouville Fractional Integral)

Let $\alpha > 0$and let $f$be a locally integrable function on the interval $[0, T]$. The Riemann–Liouville fractional integral of order $\alpha$is defined as

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, t \in [0, T],$$

where $\Gamma(\cdot)$denotes the Gamma function.

Definition 2.2 (Caputo Fractional Derivative)

Let $n-1 < \alpha < n$, where $n \in \mathbb{N}$, and let $f \in C^n([0, T])$. The Caputo fractional derivative of order $\alpha$is defined by

$$(D_C^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds.$$

The Caputo derivative is particularly useful in applied problems because it allows the formulation of initial conditions in terms of classical integer-order derivatives, making it more consistent with physical interpretations.

2.2 Fractional Differential Equations

Consider a nonlinear fractional-order differential equation of the form

$$D_C^{\alpha} x(t) = f(t, x(t)), t \in [0, T], 0 < \alpha < 1,$$

subject to the initial condition

$$x(0) = x_0.$$

Using properties of the Caputo derivative, the above problem can be transformed into an equivalent integral equation:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds.$$

This integral formulation plays a crucial role in the application of fixed point theorems, as it allows the problem to be expressed in an operator-theoretic framework.

2.3 Function Spaces

Let $C([0,T],\mathbb{R})$denote the space of all continuous real-valued functions defined on $[0,T]$. This space, equipped with the supremum norm

$$\| x \|_\infty = \sup_{t\in[0,T]} | x(t) |,$$

forms a **Banach space**.

Throughout this paper, solutions of the fractional differential equation are sought in this Banach space. The completeness of $C([0,T],\mathbb{R})$ensures the applicability of fixed-point results.

2.4 Fixed Point Theory

Fixed point theory provides powerful tools for analyzing nonlinear equations by studying points that remain invariant under a given mapping.

Definition 2.3 (Fixed Point)

Let Xbe a nonempty set and let $T: X \to X$be a mapping. A point $x \in X$is called a fixed point of Tif

$$T(x) = x.$$

Theorem 2.1 (Banach Contraction Principle)

Let (X, d)be a complete metric space and let $T: X \to X$be a contraction mapping, i.e., there exists a constant $0 < k < 1$such that $d(Tx, Ty) \le k\, d(x, y)$for all $x, y \in X$.

Then Thas a unique fixed point in X.

The Banach contraction principle is particularly useful for proving **existence and uniqueness** results, as the contraction condition guarantees both properties simultaneously.

Theorem 2.2 (Schauder Fixed Point Theorem)

Let Xbe a Banach space and let $K \subset X$be a nonempty, closed, bounded, and convex set. If $T: K \to K$is a continuous and compact operator, then Thas at least one fixed point in K.

Schauder's theorem is commonly used when contraction conditions are not satisfied but compactness properties can be established. Unlike Banach's principle, Schauder's theorem guarantees existence but not uniqueness.

2.5 Operator Formulation

Define an operator $\mathcal{T}: C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$by

$$(\mathcal{T}x)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds.$$

The problem of finding a solution to the fractional differential equation is equivalent to finding a fixed point of the operator $\mathcal{T}$. This equivalence allows the use of fixed-point theorems to establish existence and uniqueness results.

3. MAIN RESULTS

In this section, we establish the main existence and uniqueness results for nonlinear fractional-order differential equations using fixed point techniques. The analysis is carried out in an appropriate Banach space framework, and the results are derived under suitable assumptions on the nonlinear function.

We consider the nonlinear fractional-order initial value problem

$$D_C^\alpha x(t) = f(t, x(t)), t \in [0,T], 0 < \alpha < 1,$$

with the initial condition

$$x(0) = x_0,$$

where $f: [0,T] \times \mathbb{R} \to \mathbb{R}$is a given continuous function.

As shown in Section 2, this problem is equivalent to the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds.$$

The main objective is to prove the existence and uniqueness of solutions of this integral equation by applying fixed point theorems.

3.1 Assumptions

To establish the main results, we impose the following standard assumptions on the nonlinear function f:

(A1) The function $f(t,x)$is continuous on $[0,T] \times \mathbb{R}$.

(A2) There exists a constant $L > 0$ such that

$$| f(t,x) - f(t,y) | \leq L | x - y | \text{ for all } t \in [0,T],\ x,y \in \mathbb{R}.$$

That is, f satisfies a Lipschitz condition with respect to the second variable.

(A3) There exists a constant $M > 0$ such that

$$| f(t,x) | \leq M \text{ for all } t \in [0,T],\ x \in \mathbb{R}.$$

These assumptions are commonly used in the theory of fractional differential equations and ensure the applicability of fixed-point methods.

3.2 Existence and Uniqueness via Banach Contraction Principle

We first establish an existence and uniqueness result using Banach's contraction principle.

Theorem 3.1 (Existence and Uniqueness)

Assume that conditions **(A1)** and **(A2)** hold. If

$$\frac{LT^\alpha}{\Gamma(\alpha+1)} < 1,$$

then the nonlinear fractional-order initial value problem admits a unique solution in the Banach space $C([0,T],\mathbb{R})$.

Proof (Outline).

Define the operator $\mathcal{T}$ on $C([0,T],\mathbb{R})$ by

$$(\mathcal{T}x)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))\, ds.$$

Using assumption **(A2)** and standard estimates, one can show that

$$\| \mathcal{T}x - \mathcal{T}y \|_\infty \leq \frac{LT^\alpha}{\Gamma(\alpha+1)} \| x - y \|_\infty.$$

Hence, $\mathcal{T}$ is a contraction mapping. By Banach's contraction principle, $\mathcal{T}$ has a unique fixed point in $C([0,T],\mathbb{R})$, which corresponds to the unique solution of the problem. ■

3.3 Existence Result via Schauder Fixed Point Theorem

Next, we establish an existence result under weaker conditions using Schauder's fixed point theorem.

Theorem 3.2 (Existence of Solutions)

Assume that conditions **(A1)** and **(A3)** hold. Then the nonlinear fractional-order initial value problem has **at least one solution** in $C([0,T],\mathbb{R})$.

Proof (Outline).

Let

$$B_R = \{x \in C([0,T],\mathbb{R}) : \| x \|_\infty \leq R\},$$

where $R > 0$ is chosen suitably large. Under assumption **(A3)**, it can be shown that the operator $\mathcal{T}$ maps B_R into itself. Furthermore, $\mathcal{T}$ is continuous and compact due to the properties of the fractional integral operator. Hence, by Schauder's fixed point theorem, $\mathcal{T}$ has at least one fixed point in B_R, which corresponds to a solution of the problem.

3.4 Discussion of Results

The above results demonstrate that fixed point techniques provide a unified and effective approach for analyzing nonlinear fractional-order differential equations. Theorem 3.1 guarantees both existence and uniqueness under a Lipschitz condition, while Theorem 3.2 ensures existence under more general boundedness assumptions.

These results extend several classical existence and uniqueness theorems from ordinary differential equations to the fractional-order setting. Moreover, they highlight the role of fractional order α and the time interval T in determining the solvability of the problem.

4. PROOFS OF THE MAIN RESULTS

This section provides detailed proofs of the existence and uniqueness results stated in Section 3. The proofs rely on fundamental properties of fractional integrals, Banach space techniques, and fixed point theorems.

4.1 Proof of Theorem 3.1 (Existence and Uniqueness)

We consider the nonlinear fractional-order initial value problem

$$D_C^\alpha x(t) = f(t,x(t)), t \in [0,T], 0 < \alpha < 1,$$

with initial condition $x(0) = x_0$.

As shown earlier, this problem is equivalent to the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))\, ds.$$

Define the operator $\mathcal{T}: C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by

$$(\mathcal{T}x)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds.$$

We aim to show that $\mathcal{T}$ is a contraction mapping under the assumptions of Theorem 3.1.
Let $x, y \in C([0,T], \mathbb{R})$. Then, for each $t \in [0,T]$,

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, x(s)) - f(s, y(s))]\, ds \right|.$$

Using the Lipschitz condition **(A2)**, we obtain

$$|f(s, x(s)) - f(s, y(s))| \leq L\, |x(s) - y(s)|.$$

Hence,

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)|\, ds.$$

Taking the supremum over $t \in [0,T]$ and using the definition of the supremum norm, we have

$$\| \mathcal{T}x - \mathcal{T}y \|_\infty \leq \frac{L}{\Gamma(\alpha)} \| x - y \|_\infty \int_0^T (T-s)^{\alpha-1} ds.$$

Since

$$\int_0^T (T-s)^{\alpha-1} ds = \frac{T^\alpha}{\alpha} = \frac{T^\alpha}{\Gamma(\alpha+1)} \Gamma(\alpha),$$

it follows that

$$\| \mathcal{T}x - \mathcal{T}y \|_\infty \leq \frac{LT^\alpha}{\Gamma(\alpha+1)} \| x - y \|_\infty.$$

By the assumption

$$\frac{LT^\alpha}{\Gamma(\alpha+1)} < 1,$$

the operator $\mathcal{T}$ is a contraction mapping on the Banach space $C([0,T], \mathbb{R})$. Therefore, by Banach's contraction principle, $\mathcal{T}$ admits a unique fixed point $x^* \in C([0,T], \mathbb{R})$.
This fixed point satisfies $\mathcal{T}x^* = x^*$, which implies that x^* is the unique solution of the original nonlinear fractional-order differential equation. This completes the proof of Theorem 3.1.

4.2 Proof of Theorem 3.2 (Existence of Solutions)
We now prove the existence of solutions using Schauder's fixed point theorem under assumptions **(A1)** and **(A3)**.
Let

$$B_R = \{x \in C([0,T], \mathbb{R}) : \| x \|_\infty \leq R\},$$

where $R > 0$ is a positive constant to be determined.
Step 1: $\boldsymbol{\mathcal{T}(B_R) \subset B_R}$
For any $x \in B_R$ and $t \in [0,T]$,

$$|(\mathcal{T}x)(t)| \leq |x_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s))|\, ds.$$

Using assumption **(A3)**, we have $|f(s, x(s))| \leq M$. Thus,

$$|(\mathcal{T}x)(t)| \leq |x_0| + \frac{M}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds = |x_0| + \frac{MT^\alpha}{\Gamma(\alpha+1)}.$$

Choose

$$R \geq |x_0| + \frac{MT^\alpha}{\Gamma(\alpha+1)}.$$

Then $\| \mathcal{T}x \|_\infty \leq R$, implying that $\mathcal{T}(B_R) \subset B_R$.
Step 2: Continuity of $\boldsymbol{\mathcal{T}}$
Let $x_n \to x$ in $C([0,T], \mathbb{R})$. Since f is continuous and bounded, the dominated convergence theorem implies that

$$\mathcal{T}x_n(t) \to \mathcal{T}x(t)$$

uniformly on $[0, T]$. Hence, $\mathcal{T}$ is continuous.
Step 3: Compactness of $\boldsymbol{\mathcal{T}}$
The operator $\mathcal{T}$ maps bounded sets into equicontinuous and uniformly bounded sets due to the smoothing property of the fractional integral. By the Arzelà–Ascoli theorem, $\mathcal{T}$ is compact.
Since B_R is nonempty, closed, bounded, and convex, and $\mathcal{T}$ is continuous and compact with $\mathcal{T}(B_R) \subset B_R$, Schauder's fixed point theorem guarantees the existence of at least one fixed point of $\mathcal{T}$ in B_R.
Thus, the nonlinear fractional-order differential equation admits at least one solution in $C([0,T], \mathbb{R})$.
This completes the proof of Theorem 3.2.

5. ILLUSTRATIVE EXAMPLE

In this section, we present an illustrative example to demonstrate the applicability of the existence and uniqueness results established in the previous sections. The example verifies the assumptions of Theorems 3.1 and 3.2 and confirms the effectiveness of the fixed-point approach for nonlinear fractional-order differential equations.

Example 5.1

Consider the nonlinear fractional-order initial value problem

$$D_C^{\alpha} x(t) = \lambda \sin(x(t)), t \in [0, T], 0 < \alpha < 1,$$

subject to the initial condition

$$x(0) = x_0,$$

where $\lambda > 0$ is a real constant.

Verification of Assumptions

We define the nonlinear function

$$f(t, x) = \lambda \sin(x).$$

Continuity

The function $f(t, x)$ is continuous with respect to both variables on $[0, T] \times \mathbb{R}$, since the sine function is continuous on $\mathbb{R}$. Hence, assumption **(A1)** is satisfied.

Lipschitz Condition

For any $x, y \in \mathbb{R}$,

$$| f(t, x) - f(t, y) | = \lambda | \sin(x) - \sin(y) | \leq \lambda | x - y |.$$

Thus, f satisfies a Lipschitz condition with Lipschitz constant $L = \lambda$. Therefore, assumption **(A2)** holds.

Boundedness

Since $| \sin(x) | \leq 1$ for all $x \in \mathbb{R}$, we have

$$| f(t, x) | = \lambda | \sin(x) | \leq \lambda.$$

Hence, assumption **(A3)** is satisfied with $M = \lambda$.

Existence and Uniqueness Result

From Theorem 3.1, the problem admits a **unique solution** in $C([0, T], \mathbb{R})$ provided that

$$\frac{\lambda T^{\alpha}}{\Gamma(\alpha + 1)} < 1.$$

This condition establishes a clear relationship between the fractional order α, the time interval T, and the parameter λ. When this inequality holds, the operator associated with the integral formulation of the problem becomes a contraction, ensuring uniqueness.

Integral Formulation

The equivalent integral equation corresponding to the given fractional differential equation is

$$x(t) = x_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \sin(x(s)) \, ds.$$

Define the operator $\mathcal{T}: C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ by

$$(\mathcal{T}x)(t) = x_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \sin(x(s)) \, ds.$$

As shown earlier, this operator satisfies all the conditions of Banach's contraction principle under the stated inequality. Hence, the fixed point of $\mathcal{T}$ corresponds to the unique solution of the problem.

Discussion

This example illustrates how the abstract existence and uniqueness results can be applied to a concrete nonlinear fractional-order differential equation. The choice of a trigonometric nonlinearity highlights the flexibility of the theoretical framework, as the results remain valid even for nonlinearities that are not polynomial in nature.

Moreover, the condition

$$\frac{\lambda T^{\alpha}}{\Gamma(\alpha + 1)} < 1$$

clearly demonstrates the influence of the fractional order and the length of the time interval on the solvability of the problem. Smaller values of α or T increase the likelihood of satisfying the contraction condition, reflecting the inherent nonlocal effects of fractional derivatives.

6. CONCLUSION

In this paper, we have investigated the existence and uniqueness of solutions for a class of nonlinear fractional-order differential equations using fixed point techniques. Fractional-order models, due to their inherent nonlocality and memory effects, provide a more realistic mathematical framework for describing complex phenomena than classical integer-order differential equations. However, this same nonlocal nature introduces significant analytical challenges, particularly in establishing the well-posedness of the corresponding initial value problems.

By transforming the nonlinear fractional differential equation into an equivalent integral equation, we formulated the problem within an operator-theoretic framework in a suitable Banach space. This approach enabled the effective application of fixed-point theorems from functional analysis. Two complementary fixed-point techniques were employed in this study. The Banach contraction principle was used to establish sufficient conditions for both existence and uniqueness of solutions, while Schauder's fixed point theorem was applied to guarantee existence under weaker assumptions.

The main results demonstrate that if the nonlinear term satisfies a Lipschitz condition with respect to the unknown function and an appropriate contraction condition involving the fractional order and the time interval holds, then the fractional-order problem admits a unique solution. On the other hand, even in the absence of a Lipschitz condition, the existence of at least one solution can be ensured under suitable boundedness and continuity assumptions. These results highlight the versatility of fixed-point methods in handling nonlinear fractional-order differential equations.

An illustrative example involving a nonlinear trigonometric term was presented to validate the theoretical findings. The example clearly shows how the abstract assumptions can be verified in practice and how the fractional order and model parameters influence the solvability conditions. This reinforces the applicability of the theoretical framework to a wide range of nonlinear fractional-order models.

The contributions of this work are twofold. From a theoretical perspective, the results extend classical existence and uniqueness theorems from ordinary differential equations to the fractional-order setting and provide a unified analytical framework for nonlinear problems. From an applied perspective, the findings offer a solid mathematical foundation for fractional-order models arising in physics, biology, engineering, and related disciplines, where ensuring the existence and uniqueness of solutions is essential for the reliability of simulations and interpretations.

Several directions for future research naturally arise from this study. The analysis can be extended to systems of nonlinear fractional-order differential equations and to problems involving different types of fractional derivatives, such as the Hilfer or Atangana–Baleanu derivatives. Furthermore, stability analysis, dependence on parameters, and numerical approximation schemes can be investigated within the same functional analytic framework. Such extensions would further strengthen the theoretical understanding and practical applicability of nonlinear fractional-order differential equations.

In conclusion, fixed point theory provides a powerful and flexible approach for analyzing nonlinear fractional-order differential equations. The results obtained in this paper contribute to the growing body of literature on fractional calculus and serve as a useful reference for future theoretical and applied investigations in this rapidly evolving field.

7. REFERENCES

[1] Podlubny, I. (1999). Fractional differential equations. Academic Press.
[2] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations. Elsevier.
[3] Diethelm, K. (2010). The analysis of fractional differential equations. Springer.
[4] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). Fractional integrals and derivatives: Theory and applications. Gordon and Breach Science Publishers.
[5] Lakshmikantham, V., Leela, S., & Devi, J. V. (2009). Theory of fractional dynamic systems. Cambridge Scientific Publishers.
[6] Zhou, Y. (2014). Basic theory of fractional differential equations. World Scientific.
[7] Krasnoselskii, M. A. (1964). Topological methods in the theory of nonlinear integral equations. Pergamon Press.
[8] Zeidler, E. (1986). Nonlinear functional analysis and its applications I: Fixed point theorems. Springer.
[9] Burton, T. A. (2005). Stability and periodic solutions of ordinary and functional differential equations. Dover Publications.
[10] Miller, K. S., & Ross, B. (1993). An introduction to the fractional calculus and fractional differential equations. Wiley.
[11] Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, J. J. (2012). Fractional calculus: Models and numerical methods. World Scientific.

[12] Agarwal, R. P., Benchohra, M., & Hamani, S. (2008). Boundary value problems for differential equations with fractional order. Advances in Difference Equations, 2008, Article 623529. https://doi.org/10.1155/2008/623529
[13] Almeida, R., Bastos, N. R. O., & Monteiro, M. T. T. (2016). Modeling some real phenomena by fractional differential equations. Mathematical Methods in the Applied Sciences, 39(16), 4846–4855. https://doi.org/10.1002/mma.3898
[14] Goodrich, C. S., & Peterson, A. C. (2015). Discrete fractional calculus. Springer.
[15] Smart, D. R. (1974). Fixed point theorems. Cambridge University Press.